INTEGRAL TRANSFORMS OF *K*-WEYL FRACTIONAL INTEGRALS

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ABSTRACT: In this paper, we find Mellin, Laplace, Fourier and generalized Stieltjes transforms of k - Weyl fractional integral. When $k \to 1$, these results hold true for the usual Weyl fractional integral.

Mathematics Subject Classification: 26A33, 42A38
KEYWORDS: Mellin, Laplace, Fourier and generalized Stieltjes Transforms, k -Weyl fractional integral

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1. INTRODUCTION

There are many initial and boundary value problems in applied mathematics, physics and engineering which can be solved by the use of integral transforms. These transforms are very useful for solving differential as well as integral equations.

Diaz and Pariguan [1] have defined new functions called k -gamma and k -beta functions and the Pochhammer k -symbol that is generalization of the classical gamma and beta functions and the classical Pochhammer symbol.

Mubeen and Habibullah [2] have introduced the k -Riemann-Liouville fractional integral. They defined the k -Riemann-Liouville fractional integral by using the k -Gamma function (the generalized form of the classical Gamma function). Romero and Luque [3] defined k -Weyl fractional integral by using classical convolution (*).

Definition 1. The k -gamma function is defined as

$$\Gamma_k(x) = \int_0^\infty e^{\frac{t^k}{k}} t^{x-1} dt, \ \ \mathrm{Re}(x) > 0 \qquad (1)$$

and $\Gamma(x) = \lim_{k\to 1} \Gamma_k(x), \Gamma_k(x) = k^{\frac{x}{k}-1} \Gamma(\frac{x}{k}).$

Definition 2. The k -beta function is defined as

$$B_k(x,y) = \frac{1}{k}\int_0^1 (1-t)^{\frac{y}{k}-1} t^{\frac{x}{k}-1)} dt; \qquad (2)$$

$$\mathrm{Re}(x) > 0, \mathrm{Re}(y) > 0$$

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and $B_k(x,y) = \dfrac{\Gamma_k(x)\Gamma_k(y)}{\Gamma_k(x+y)}.$

Definition 3. Let α be a real number, $0 < \alpha < 1, k > 0$. The k -Weyl fractional integral is defined by

$$W_k^{\alpha}(f(x)) = \frac{1}{k\Gamma_k(\alpha)} \int_x^\infty (t-x)^{\frac{\alpha}{k}-1} f(t)dt, x \ge 0, t > 0.$$

(3).

Definition 4. The Mellin transform of a real scalar function $f(x)$ is defined as

$$f^*(s) = M[f(x)] = \int_0^\infty x^{s-1} f(x) dx, \qquad (4).$$

whenever $f^*(s)$ exists. It is a function of the arbitrary parameter $s \in \mathbb{C}, \mathrm{Re}(s) > 0.$

Definition 5. The Laplace transform of a real scalar piecewise continuous function $f(x)$ with parameter s is defined as

$$F(s) = L[f(u)] = \int_0^\infty e^{-su} f(u) du. \qquad (5).$$

whenever $F(s)$ exists. It is a function of the arbitrary parameter $s \in \mathbb{C}, u \in \mathbb{R}^+, \mathrm{Re}(s) > 0.$

Definition 6: The Fourier transform of function $f(x)$ of a real variable x is defined as

$$G(t) = F[f(x)] = \int_{-\infty}^{\infty} e^{-\iota xt} f(x) dx, \qquad (6).$$

whenever $G(t)$ exists. It is a function of the real variable t.

Definition 7: Let the function $f(x) \in S(\mathbb{R})$ the Schwartzian space of functions that decay rapidly at infinity together with all derivatives, then the generalized Stieltjes transform is defined as

$$S_\beta[f(x)] = \int_0^\infty (x+y)^{-\beta} f(x) dx, y > 0, \beta > 0. \ (7).$$

1.1 Theorem: Let f be continuous on $[0,\infty)$ and let $\alpha \in (0,1), k > 0$. Then for $x \ge 0$

$$M\left[W_k^{\alpha}(f(x))\right] = \frac{\Gamma(sk)}{\Gamma(sk+\alpha)} f^*(s + \frac{\alpha}{k}), \ \mathrm{Re}(s) > 0. \qquad (8).$$

Proof: Using (5), we obtain

$$M\left[W_k^{\alpha} f(x)\right]=\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty} x^{s-1}[\int_x^{\infty}(t-x)^{\frac{\alpha}{k}-1} f(t)dt]dx.$$

Using Fubini's theorem

$$M\left[W_k^{\alpha} f(x)\right]=\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty} f(t)[\int_0^{t} x^{s-1}(t-x)^{\frac{\alpha}{k}-1}dx]dt.$$

By substituting $x=ty$

$$M\left[W_k^{\alpha} f(x)\right]=\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty} t^{\frac{sk+\alpha}{k}-1} f(t)\left[\int_0^{1}(1-y)^{\frac{\alpha}{k}-1}(y)^{\frac{sk}{k}-1}dy\right]dt.$$

Using (2) and (4), we get (8).

1.2 Theorem: Let f be continuous on $[0,\infty)$ and let $\alpha\in(0,1), k>0.$ Then for $x\geq 0$

$$M\left[W_k^{\alpha}(x^{-\frac{\alpha}{k}} f(x))\right]=\frac{\Gamma(sk)}{\Gamma(sk+\alpha)} f^{*}(s),\ \mathrm{Re}(s)>0. \quad (9).$$

Proof: Using (4)

$$M\left[W_k^{\alpha} f(x)\right]=\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty} x^{s-1}[\int_x^{\infty}(t-x)^{\frac{\alpha}{k}-1} f(t)dt]dx.$$

By Fubini's theorem

$$M\left[W_k^{\alpha} f(x)\right]=\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty} f(t)[\int_0^{t} x^{s-1}(t-x)^{\frac{\alpha}{k}-1}dx]dt.$$

By substituting $x=ty$

$$M\left[W_k^{\alpha}(x^{-\frac{\alpha}{k}} f(x))\right]=\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty} t^{\frac{sk}{k}-1} f(t)\left[\int_0^{1}(1-y)^{\frac{\alpha}{k}-1}(y)^{\frac{sk}{k}-1}dy\right]dt.$$

Using (2) and (4) we get (9).

Example:1 Let $f(x)=e^{-x}$ then using (9), we get

$$M\left[W_k^{\alpha}(x^{-\frac{\alpha}{k}} e^{-x})\right]=\frac{\Gamma(sk)\Gamma(x)}{\Gamma(sk+\alpha)}.$$

1.3 Theorem: Let f be continuous on $[0,\infty)$ and let $\alpha\in(0,1), k>0, \beta\in\mathbb{R}$. Then for $x\geq 0$

$$L\left[W_k^{\alpha}(e^{\beta x} f(x))\right]=(-sk)^{-\frac{\alpha}{k}} F(s-\beta).$$

$$s>\beta,\ \mathrm{Re}(s)>0. \quad (10).$$

Proof: Using (5)

$$L\left[W_k^{\alpha}(e^{\beta x} f(x))\right]=\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty} e^{-su}[\int_u^{\infty}(t-u)^{\frac{\alpha}{k}-1} e^{\beta t} f(t)dt]du.$$

Using Fubini's theorem

$$L\left[W_k^{\alpha}(e^{\beta x} f(x))\right]=\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty} e^{\beta t} f(t)[\int_u^{\infty}(t-u)^{\frac{\alpha}{k}-1} e^{-su}du]dt.$$

By substituting $t-u=y$

$$L\left[W_k^{\alpha}(e^{\beta x} f(x))\right]=\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty} e^{-st+\beta t} f(t)\left[\int_0^{\infty}(y)^{\frac{\alpha}{k}-1}(e)^{sy}dy\right]dt.$$

By substituting $-sy=\tau$

$$L\left[W_k^\alpha(e^{\beta x}f(x))\right]=$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_0^\infty e^{-st+\beta t}f(t)(-s)^{-\frac{\alpha}{k}}\left[\int_0^\infty(\tau)^{\frac{\alpha}{k}-1}(e)^{-\tau}d\tau\right]dt.$$

Using (1) and (6), we get (10).

1.4 Theorem: Let f be continuous on $[0,\infty)$ and let $\alpha\in(0,1),k>0.$ Then for $x\geq 0$

$$L\left[W_k^\alpha f(x)\right]=(-sk)^{-\frac{\alpha}{k}}F(s),\ \mathrm{Re}(s)>0. \quad (11).$$

Proof: Substituting $\beta=0$ in (10), we get (11).

Also see Romero and Luque [4].

Example 2. Let $f(x)=x^n, n\in\mathbb{N}^+.$ Let $\alpha\in(0,1),k>0.$ Then for $x\geq 0$ using (11), we get

$$L\left[W_k^\alpha f(x)\right]=(-sk)^{-\frac{\alpha}{k}}\frac{n!}{s^{n+1}}.$$

1.5 Theorem: Let $f(x)\in L_1(\mathbb{R})$ and let $\alpha\in(0,1),k>0.$ Then for $x\geq 0$

$$F\left[W_k^\alpha(f(x))\right]=\frac{1}{(-\imath tk)^{\frac{\alpha}{k}}}G(t),\ 0<\mathrm{Re}(\alpha)<1. (12).$$

Proof: Using (7)

$$F\left[W_k^\alpha f(x)\right]=$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_{-\infty}^\infty e^{-\imath xt}[\int_x^\infty(u-x)^{\frac{\alpha}{k}-1}f(u)du]dx.$$

Using Fubini's theorem

$$F\left[W_k^\alpha f(x)\right]=$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_{-\infty}^\infty f(u)[\int_u^\infty e^{-\imath xt}(u-x)^{\frac{\alpha}{k}-1}dx]du.$$

By substituting $u-x=y$

$$F\left[W_k^\alpha f(x)\right]=$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_{-\infty}^\infty(e)^{-\imath ut}f(u)\left[\int_0^\infty(y)^{\frac{\alpha}{k}-1}(e)^{\imath ty}dy\right]du.$$

Substituting $-\imath ty=\tau$

$$F\left[W_k^\alpha f(x)\right]=$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_{-\infty}^\infty(e)^{-\imath ut}f(u)\left[\int_0^\infty(y)^{\frac{\alpha}{k}-1}(e)^{\imath ty}dy\right]du.$$

Using (1) and (5) we get (12).

1.6 Theorem: Let $f(x)\in L_1(\mathbb{R})$ and let $\alpha\in(0,1),k>0,\beta\in\mathbb{R}$. Then for $x\geq 0$

$$F\left[W_k^\alpha(e^{-\imath\beta x}f(x))\right]=\frac{1}{(-\imath tk)^{\frac{\alpha}{k}}}G(\beta+t). \quad (13).$$

Proof: Using (7)

$$F\left[W_k^\alpha(e^{-\imath\beta x}f(x))\right]=$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_{-\infty}^\infty e^{-\imath xt}[\int_x^\infty(u-x)^{\frac{\alpha}{k}-1}e^{-\imath\beta u}f(u)du]dx.$$

Using Fubini's theorem

$$F\left[W_k^\alpha(e^{-\imath\beta x}f(x))\right]=$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_{-\infty}^\infty e^{-\imath\beta u}f(u)[\int_u^\infty e^{-\imath xt}(u-x)^{\frac{\alpha}{k}-1}dx]du.$$

Substituting $u-x=y$

$$F\left[W_k^\alpha(e^{-\imath\beta x}f(x))\right]=$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_{-\infty}^\infty e^{-\imath u(\beta+t)}f(u)\left[\int_0^\infty(y)^{\frac{\alpha}{k}-1}(e)^{\imath ty}dy\right]du.$$

Substituting $-\imath ty=\tau$

$$F\left[W_k^{\alpha}(e^{-\iota\beta x}f(x))\right]=$$

$$\frac{(-\iota k)^{-\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_{-\infty}^{\infty}e^{-\iota u(\beta+t)}f(u)\left[\int_0^{\infty}(\tau)^{\frac{\alpha}{k}-1}(e)^{-\tau}d\tau\right]du.$$

Using (1) and (5) we get (13).

1.7 Theorem: Let f be continuous on $[0,\infty)$ and let $\alpha\in(0,1),k>0,\beta\in\mathbb{R}$. Then for $x\geq 0$

$$S_\beta\left[W_k^{\alpha}(x+y)^{-\lambda}f(x)\right]=$$

$$\frac{(x+y)^{\frac{\alpha}{k}}\Gamma_k(\beta k)}{\Gamma_k(\alpha+\beta k)}S_{\beta+\lambda}\left[f(x)\right],y>0,\mathrm{Re}(x)>0,\lambda>0. \quad (14).$$

Proof: Using (8)

$$S_\beta\left[W_k^{\alpha}(x+y)^{-\lambda}f(x)\right]=$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty}(x+y)^{-\beta}[\int_x^{\infty}(t-x)^{\frac{\alpha}{k}-1}(t+y)^{-\lambda}f(t)dt]dx.$$

Using Fubini's theorem

$$S_\beta\left[W_k^{\alpha}(x+y)^{-\lambda}f(x)\right]=$$

$$\frac{1}{k\Gamma_k(\alpha)}\int_0^{\infty}(t+y)^{-\lambda}f(t)[\int_t^{\infty}(t-x)^{\frac{\alpha}{k}-1}(x+y)^{-\beta}dx]dt.$$

By substituting $u=\dfrac{t-x}{y+x}$

$$S_\beta\left[W_k^{\alpha}(x+y)^{-\lambda}f(x)\right]=$$

$$\frac{(x+y)^{\frac{\alpha}{k}}}{k\Gamma_k(\alpha)}\int_0^{\infty}(y+t)^{-(\beta+\lambda)}f(t)[\int_0^{1}(u)^{\frac{\alpha}{k}-1}(1-u)^{\beta-1}du]dt.$$

Using (2) and (6), we get (14).

1.8 Theorem: Let f be continuous on $[0,\infty)$ and let $\alpha\in(0,1),k>0,\beta\in\mathbb{R}$. Then for all $x\geq 0$

$$S_\beta\left[W_k^{\alpha}(f(x))\right]=$$

$$\frac{(x+y)^{\frac{\alpha}{k}}\Gamma_k(\beta k)}{\Gamma_k(\alpha+\beta k)}S_\beta\left[f(x)\right],\mathrm{Re}(x)>0,y>0. \quad (15).$$

Proof: Substituting $\lambda=0$ in (14), we get (15).

Also see Romero and Luque [4].

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